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ANTIPLANE DEFORMATION OF A PERIODICALLY LAYERED COMPOSITE WITH A CRACK. A NON-HOMOGENIZATION APPROACH

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Abstract—Fracture behaviour of an infinite periodically layered composite body, with a Mode 3 crack parallel to the layering, is investigated. Upon deriving the Green function for a dislocation in a layered space, the problem is reduced to a singular integral equation of the first kind. The specific case of the bimaterial multilayered composite is considered in detail. The numerical results obtained for the stress intensity factor allows one to estimate the accuracy of the known approximate models with a reduced number of layers. It is found that for certain parameter combinations the stress intensity factor for the interface crack may exceed the corresponding value for a crack in a homogeneous space. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Employing new composite materials consisting of a large number of dissimilar layers requires the study of different aspects of their fracture behavior. One of the important cases of interest is fracture initiated by a single flaw parallel to the interfaces which is usually modeled as a crack. If the size of the crack is not too large with respect to the characteristic layer thickness, it is plausible to assume that the perturbation in the stress state, caused by the crack, is localized in the two or three nearby layers. Consequently, only these layers are to be considered in the analysis of the problem. Solutions of numerous problems of this type are known in the literature.

In a more general situation, when the above mentioned assumption is not valid, models with a large number of layers are to be employed. Far fewer problems of this type have been investigated. Erdogan and Gupta (1971) presented a general formulation of the problem of a crack in a composite consisting of an arbitrary finite number n of layers and solved some in-plane and out of plane problems for $n = 3$. Chen and Sih (1971) considered antiplane deformation of a four layered composite with a crack assuming that the elastic properties of the outer layers of infinite thickness are the averages over a large number of layers of the actual laminate. If the size of a crack in a composite consisting of a multiple repeating group of layers is small with respect to characteristic size of the body, then the model of a crack in a periodically laminated space is valid. In the framework of this model, Kaczynski and Matysiak (1988) solved the problem of an arbitrary loaded crack in a periodically layered bimaterial composite. To obtain the result they employed a specific homogenization procedure. Later, this approach was extended by the authors (Kaczynsky and Matysiak, 1989) to the case of a system of interface cracks and by Kaczynski *et al.* (1994) to the case of a periodically layered composite of finite thickness. The description of another homogenization method for the solution of problems on periodically layered composites with cracks may be found in Parton and Kudryavtsev (1993). The problem of the semi-infinite Mode 3 crack in a periodically laminated composite is considered by Ryvkin (1996). The solution which is obtained in closed form does not depend on any homogenization procedure.

The subject of the present paper is a finite Mode 3 crack in a composite consisting of an infinite number of periodically arranged layers. The method of solution does not rely on any averaging or homogenization procedure. It is based on the combination of the dislocation approach developed by Erdogan and Gupta (1971) and the representative cell method suggested by Nuller and Ryvkin (1980). In the next section the mathematical analysis of the problem is presented. After deriving the closed form expression of the Green function for a single dislocation in the periodically layered space, the problem is reduced to a singular integral equation of the first kind which is solved numerically. In Section 3 the numerical and analytical results for the stress intensity factor in the specific case of a periodically layered bimaterial composite are presented and discussed.

2. ANALYSIS

2.1. *Problem formulation*

Consider a composite body consisting of an infinite number of periodically repeating isotropic elastic layers of three different types, with thicknesses h_r, $r = 1, 2, 3$. The materials of the layers are characterized by the elastic shear moduli μ_r (Fig. 1). Let (x', y', z') be the system of global Cartesian coordinates with the origin at one of the interfaces and y'-axis perpendicular to the layering. The geometrical periodicity of the body is violated by an interface crack occupying the region

$$-a < x' < a, \quad \infty < z' < \infty \tag{1}$$

on the plane $y' = 0$. The uniformly distributed shear loading p_0 applied to the crack faces in the z'-direction produces an antiplane stress state which will be investigated.

The body is considered as an assemblage of an infinite number of bonded cells

$$kh - h_1 < y' < kh + h_2 + h_3, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

where $h = h_1 + h_2 + h_3$ is the thickness of a cell. Hence, each cell contains three layers, one of each type, and the cells are identical except that the one numbered "0" (i.e., the cell occupying the region $-h_1 < y' < h_2 + h_3$) has a cut between the first and the second layers. It is convenient to formulate the problem using the local coordinate systems introduced in each cell in the following manner:

$$x \equiv x', \quad y = y' - kh, \quad z \equiv z'. \tag{3}$$

The antiplane stress strain in the r-th layer of the k-th cell is defined by the displacements in the z-direction $w_r^{(k)}(x, y)$ and the corresponding shear stresses

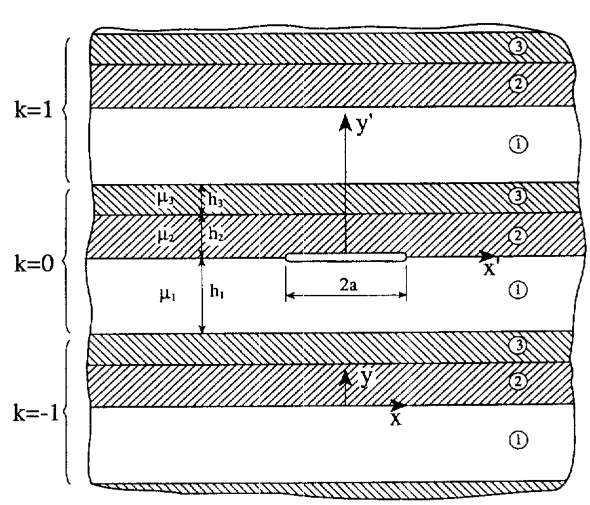

Fig. 1. Three-material periodically layered composite with a crack.

$$\tau_{r_{xz}}^{(k)} = \mu_r \frac{\partial w_r^{(k)}}{\partial x}, \quad \tau_{r_{yz}}^{(k)} = \mu_r \frac{\partial w_r^{(k)}}{\partial y}, \tag{4}$$

where $r = 1, 2, 3$; $k = 0, \pm 1, \pm 2, \ldots$ For the sake of brevity the subscript yz will hereafter be omitted: $\tau_{r_{yz}}^{(k)} \equiv \tau_r^{(k)}$. The equilibrium conditions require that the displacements in each layer are expressed by a harmonic function

$$\frac{\partial^2 w_r^{(k)}}{\partial x^2} + \frac{\partial^2 w_r^{(k)}}{\partial y^2} = 0. \tag{5}$$

Let $U_r^{(k)}(x,y) = \{w_r^{(k)}(x,y), \tau_r^{(k)}(x,y)\}$ be a vector of the boundary values of the displacements and the stresses at the planes perpendicular to the y-axis in the r-th layer. Then the bonding conditions between the cells can be written as follows:

$$U_3^{(k)}(x, h_2+h_3) - U_1^{(k+1)}(x, -h_1) = 0, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{6}$$

The same continuity conditions are to be satisfied at the inner undamaged interfaces of the cells

$$U_3^{(k)}(x, h_2) - U_2^{(k)}(x, h_2) = 0, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{7}$$

$$U_2^{(k)}(x, 0) - U_1^{(k)}(x, 0) = 0, \quad k = \pm 1, \pm 2, \ldots. \tag{8}$$

while the mixed boundary conditions in the crack plane in the cell with $k = 0$ are given by

$$\left.\begin{array}{r} w_2^{(0)}(x,0) - w_1^{(0)}(x,0) = 0 \\ \tau_2^{(0)}(x,0) - \tau_1^{(0)}(x,0) = 0 \end{array}\right\} \quad |x| > a, \tag{9}$$

$$\tau_2^{(0)}(x,0) = \tau_1^{(0)}(x,0) = -p_0, \quad |x| \leqslant a. \tag{10}$$

Using the symmetry of the elastic domain and the loading with respect to the plane $x' = 0$, it is possible to consider further only the half-space $x' > 0$, with the added condition

$$\frac{\partial w_r^{(k)}(0,y)}{\partial x} = 0, \quad r = 1,2,3; \quad k = \pm 1, \pm 2, \ldots. \tag{11}$$

The formulated boundary value problem (4)–(11) will be reduced to the singular integral equation by the method employed in Erdogan and Gupta (1971). The crack is viewed as distributed dislocations with some unknown density $f(t)$, $0 < t < a$. Consequently, the displacements are presented by the use of the Green function of the problem $\hat{w}_r^{(k)}(x, y, t)$

$$w_r^{(k)}(x,y) = \int_0^a \hat{w}_r^{(k)}(x,y,t)\,\mathrm{d}t. \tag{12}$$

This function is given by the solution of a problem on the same body without the crack and with a single dislocation at the point $x = t$, $y = 0$ in cell number 0

$$\frac{\partial}{\partial x}[\hat{w}_2^{(0)}(x,0) - \hat{w}_1^{(0)}(x,0)] = f(t)\delta(x-t), \quad 0 < x < \infty, \tag{13}$$

where $\delta(x)$ is the delta-function. Upon solving this problem the stresses $\hat{\tau}_r^{(k)}(x,y,t)$ as well as the displacements will be expressed through the one unknown function $f(t)$. In order to derive it one must employ condition (10) for the tractions on the crack faces:

$$-p_0 = \int_0^a \hat{\tau}_1^{(0)}(x, 0, t)\,\mathrm{d}t, \quad 0 < x < a. \tag{14}$$

This leads to the singular integral equation with respect to $f(t)$. Having defined it, one obtains the solution of the initial problem in the form (12).

2.2. *The Green function for a dislocation in a periodically layered space*

Consider the periodically layered body described in the previous subsection without the crack. The displacements $\hat{w}_r^{(k)}(x, y, t)$ and the stresses $\hat{\tau}_r^{(k)}(x, y, t)$ to be determined are generated by the dislocation (13).

The corresponding boundary value problem is defined by the relations (4)–(7), (11) with $w_r^{(k)}(x, y) \equiv \hat{w}_r^{(k)}(x, y, t)$, $\tau_r^{(k)}(x, y) \equiv \hat{\tau}_r^{(k)}(x, y, t)$, and the following condition at the interfaces between the layers of the first and the second types:

$$\hat{\tau}_2^{(k)}(x, 0, t) - \hat{\tau}_1^{(k)}(x, 0, t) = 0, \quad 0 < x < \infty, \tag{15}$$

$$\frac{\partial}{\partial x}[\hat{w}_2^{(k)}(x, 0, t) - \hat{w}_1^{(k)}(x, 0, t)] = F_k(x, t), \quad 0 < x < \infty, \tag{16}$$

where $F_k(x, t) = f(t)\delta_{0k}\delta(x-t)$, δ_{ij} is the Kronecker delta and $k = 0, \pm 1, \pm 2, \ldots$.

Following Nuller and Ryvkin (1980), this boundary value problem for the periodically layered space can be converted to the problem for a representative cell (a three layered strip) defined by

$$\hat{\tau}_{r_{xz}}^* = \mu_r \frac{\partial \hat{w}_r^*}{\partial x}, \quad \hat{\tau}_r^* = \mu_r \frac{\partial \hat{w}_r^*}{\partial y}. \tag{17}$$

$$\frac{\partial^2 \hat{w}_r^*}{\partial x^2} + \frac{\partial^2 \hat{w}_r^*}{\partial y^2} = 0, \tag{18}$$

$$\gamma \hat{U}_3^*(x, h_2 + h_3, t, \phi) - \hat{U}_1^*(x, -h_1, t, \phi) = 0, \tag{19}$$

$$\hat{U}_3^*(x, h_2, t, \phi) - \hat{U}_2^*(x, h_2, t, \phi) = 0, \tag{20}$$

$$\hat{\tau}_2^*(x, 0, t, \phi) - \hat{\tau}_1^*(x, 0, t, \phi) = 0, \tag{21}$$

$$\frac{\partial}{\partial x}[\hat{w}_2^*(x, 0, t, \phi) - \hat{w}_1^*(x, 0, t, \phi)] = f(t)\delta(x-t), \tag{22}$$

$$\frac{\partial \hat{w}_r^*(0, y, t, \phi)}{\partial x} = 0, \quad r = 1, 2, 3; \tag{23}$$

where $\gamma = e^{i\phi}$, $\hat{U}_r^*(x, y, t, \phi) = \{\hat{w}_r^*(x, y, t, \phi), \hat{\tau}_r^*(x, y, t, \phi)\}$ and the superscript * denotes the corresponding discrete Fourier transforms. For example,

$$\hat{U}_r^*(x, y, t, \phi) = \sum_{k=-\infty}^{k=\infty} \hat{U}_r^{(k)}(x, y, t)e^{ik\phi}. \tag{24}$$

Since the condition (19) contains a complex coefficient the displacement and the stress transforms $\hat{w}_r^*$ and $\hat{\tau}_r^*$ will be expressed in terms of complex valued functions. However, this does not prevent us to consider problem (17) to (23) for the representative cell as a usual boundary value problem for the three layered elastic composite. Consequently, it is possible to employ for the solution the standard method of integral transforms. Assume that the displacements in each layer of the cell are expressed by the use of the cosine Fourier transform

$$\hat{w}_r^*(\alpha, y, t, \phi) = \frac{2}{\pi}\int_0^\infty [A_{2r-1}e^{\alpha y} + A_{2r}e^{-\alpha y}]\cos\alpha x\,\mathrm{d}\alpha, \quad r = 1, 2, 3, \tag{25}$$

where $A_i = A_i(\alpha, t, \phi)$, $i = 1, 2, \ldots 6$ are some unknown functions. Then eqns (17), (18) and (23) are satisfied identically. Substituting (25) into (19)–(22) yields a system of six linear algebraic equations with respect to A_i

$$CA = R, \tag{26}$$

where

$$C = \begin{bmatrix} e^{-\alpha h_1} & e^{-\alpha h_1} & 0 & 0 & -\gamma e^{\alpha(h_2+h_3)} & -\gamma e^{-\alpha(h_2+h_3)} \\ e^{-\alpha h_1} & -e^{\alpha h_1} & 0 & 0 & -\gamma\mu_{31}e^{\alpha(h_2+h_3)} & \gamma\mu_{31}e^{-\alpha(h_2+h_3)} \\ 0 & 0 & e^{\alpha h_2} & e^{-\alpha h_2} & -e^{\alpha h_2} & -e^{\alpha h_2} \\ 0 & 0 & \mu_{21}e^{\alpha h_2} & -\mu_{21}e^{-\alpha h_2} & -\mu_{31}e^{\alpha h_2} & \mu_{31}e^{-\alpha h_2} \\ 1 & -1 & -\mu_{21} & \mu_{21} & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix} \tag{27}$$

$$A^{\mathrm{T}} = [A_1\ A_2\ A_3\ A_4\ A_5\ A_6], \tag{28}$$

$$R^{\mathrm{T}} = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \alpha^{-1}f\sin\alpha t] \tag{29}$$

$$\text{and}\quad \mu_{ij} = \frac{\mu_i}{\mu_j}. \tag{30}$$

Note that to obtain the last inhomogeneous equation in system (26) from (22) one must employ an additional condition

$$\lim_{x\to\infty}[\hat{w}_2^*(x, 0, t, \phi) - \hat{w}_1^*(x, 0, t, \phi)] = 0. \tag{31}$$

Its validity follows from the fact that layers are perfectly bonded everywhere except at the dislocation point and from the definition of the discrete Fourier transform (24).

Substituting the expressions derived from (26) for A_i in (25) and (17) yields "displacements" $\hat{w}_r^*(x, y, \phi)$ and "stresses" $\hat{\tau}_r^*(x, y, \phi)$ in the representative cell, proportional to the dislocation intensity $f(t)$. Applying then the inverse discrete Fourier transform

$$\hat{U}_r^{(k)}(x, y, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{U}_r^*(x, y, t, \phi)e^{-ik\phi}\,\mathrm{d}\phi \tag{32}$$

one obtains the closed form expressions for the displacements and the stresses in every cell of the periodically layered space. For example, the function $\hat{\tau}_1^{(k)}(x, y, t)$ is found to be

$$\hat{\tau}_1^{(k)}(x, y, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ik\phi}\frac{2}{\pi}\int_0^\infty \alpha\mu_1[A_1e^{\alpha y} - A_2e^{-\alpha y}]\cos\alpha x\,\mathrm{d}\alpha\,\mathrm{d}\phi, \tag{33}$$

where

$$A_1 = f(t)\check{\mu}_{21} b_1(\alpha,\phi)\sin\alpha t[\alpha b_0(\alpha,\phi)]^{-1}, \quad A_2 = f(t)\check{\mu}_{21} b_2(\alpha,\phi)\sin\alpha t[\alpha b_0(\alpha,\phi)]^{-1},$$

$$b_1(\alpha,\phi) = \frac{4\gamma\check{\mu}_{32}}{(1+\mu_{31})} e^{-\alpha h} + \bar{\mu}_{31} e^{-2\alpha(h_2+h_3)} - \bar{\mu}_{32} e^{-2\alpha h_2} + \bar{\mu}_{31}\bar{\mu}_{32} e^{-2\alpha h_3} - 1,$$

$$b_2(\alpha,\phi) = \frac{4\gamma\check{\mu}_{32}}{(1+\mu_{31})} e^{-\alpha h} - e^{-2\alpha h} + \bar{\mu}_{31}\bar{\mu}_{32} e^{-2\alpha(h_1+h_2)} - \bar{\mu}_{32} e^{-2\alpha(h_1+h_3)} + \bar{\mu}_{31} e^{-2\alpha h_1},$$

$$b_0(\alpha,\phi) = 16\mu_{31}(\mu_+)^{-1}\cos\phi e^{-\alpha h} - e^{-2\alpha h} + \bar{\mu}_{31}\bar{\mu}_{32} e^{-2\alpha(h_1+h_2)} - \bar{\mu}_{21}\bar{\mu}_{32} e^{-2\alpha(h_1+h_3)}$$
$$+\bar{\mu}_{21}\bar{\mu}_{31} e^{-2\alpha(h_2+h_3)} + \bar{\mu}_{21}\bar{\mu}_{31} e^{-2\alpha h_1} + \bar{\mu}_{21}\bar{\mu}_{32} e^{-2\alpha h_2} + \bar{\mu}_{31}\bar{\mu}_{32} e^{-2\alpha h_3} - 1,$$

$$\bar{\mu}_{ij} = \frac{-1+\mu_{ij}}{1+\mu_{ij}}, \quad \check{\mu}_{ij} = \frac{\mu_{ij}}{1+\mu_{ij}}, \quad \mu_+ = (1+\mu_{21})(1+\mu_{31})(1+\mu_{32}). \tag{34}$$

Using these results it is easy to formulate a system of integral equations for an antiplane problem of a three-material periodically layered composite with an arbitrary system of interface cracks or inclusions. For a two material composite it is also possible to consider imperfections parallel to the layers.

For the solution of the initial problem it is necessary to determine the Green function $\tilde{\tau}_1^{(0)}(x,0,t)$ appearing in (14). Because of the discontinuity of the expression in the right hand side of (33) for $y = 0$ we define

$$\tilde{\tau}_1^{(0)}(x,0,t) = \lim_{y\to -0} \tilde{\tau}_1^{(0)}(x,y,t). \tag{35}$$

After some manipulations one obtains

$$\tilde{\tau}_1^{(0)}(x,0,t) = \frac{2\mu_1\check{\mu}_{21} f(t)}{\pi}\left[\frac{t}{t^2-x^2} + \int_0^\infty R(\alpha)\sin\alpha t\cos\alpha x\,\mathrm{d}\alpha\right], \tag{36}$$

$$\text{with } R(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[b_1 - b_0 - b_2]b_0^{-1}\,\mathrm{d}\phi, \tag{37}$$

where the functions $b_j = b_j(\alpha,\phi)$ are defined by (34). Note that the integration (37) may be carried out analytically. The results are not presented for the sake of brevity.

2.3. *Integral equation*

The integral equation for deriving the unknown dislocation density $f(t)$, $0 < t < a$ is obtained by substituting (36) into (14). Introducing the non-dimensional coordinates

$$\bar{x} = \frac{x}{a}, \quad \bar{t} = \frac{t}{a}, \quad \bar{\alpha} = \alpha a, \tag{38}$$

and carrying out some manipulations, including symmetric continuation of the function $f(t)$ for the negative values of the argument

$$f(-\bar{t}) = -f(\bar{t}), \tag{39}$$

one obtains the singular integral equation of the first kind (for more details see Erdogan and Gupta (1971))

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(\bar{t})\,\mathrm{d}\bar{t}}{\bar{t}-\bar{x}} + \int_{-1}^{1} f(\bar{t})K(\bar{x},\bar{t})\,\mathrm{d}\bar{t} = -\frac{p_0}{\mu_1\check{\mu}_{21}}, \quad -1 < \bar{x} < 1. \tag{40}$$

Here

$$K(\bar{x},\bar{t}) = \frac{1}{\pi}\int_0^\infty R(\bar{\alpha})\sin[\bar{\alpha}(\bar{t}-\bar{x})]\,d\bar{\alpha} \tag{41}$$

is the non-singular Fredholm kernel. This is evident from the decaying and smoothness behavior of function $R(\bar{\alpha})$. In fact, from (37), (34), (38) it follows that for large $\bar{\alpha}$ the estimation

$$R(\bar{\alpha}) = 0\left[\exp\left(-\bar{\alpha}\frac{h_{\min}}{a}\right)\right], \quad \bar{\alpha}\to\infty, \tag{42}$$

with $h_{\min} = min[h_1, h_2]$ is valid. The absence of singularities in $R(\bar{\alpha})$ follows from the fact that, as it is possible to prove, the function $b_0(\alpha, \phi)$ in the denominator of the integrand in (37) has no zeros. For the limiting case $h_3 \to \infty$, with $\mu_1 = \mu_2$, the integral eqn (40) reduces to the equation obtained by Erdogan and Gupta (1971) for the problem of the cracked layer sandwiched between two half-spaces. Formulation of the boundary value problem in the form of integral eqn (40) requires also the knowledge of the value

$$\int_{-1}^{1} f(\bar{t})\,d\bar{t} = C_0. \tag{43}$$

For the problem considered, it follows from (39) that C_0 is equal to zero.

The theory of integral equations of type (40) has been extensively studied (Muskhelishvili, 1992). Employing the usual physical assumptions regarding the form of the crack near its tip allows one to present the unknown function $f(\bar{t})$ in the following form (Erdogan and Gupta, 1971):

$$f(\bar{t}) = \frac{F(\bar{t})}{(1-\bar{t}^2)^{1/2}}, \tag{44}$$

where $F(\bar{t})$ is some new unknown function bounded in the considered interval. Comparison between the different numerical procedures for the solution of the integral equation reveals that the most efficient way to determine the function $F(\bar{t})$ is to employ the method developed by Erdogan and Gupta (1972). It is assumed that the function $F(\bar{t})$ can be approximated with sufficient degree of accuracy by the truncated series

$$F(\bar{t}) = \sum_{i=0}^{n-1} a_i T_i(\bar{t}), \tag{45}$$

where $T_i(\bar{t})$ are the Chebyshev polynomials. Employing the version of the Gauss–Chebyshev integration formula for the singular integrals, obtained in the reference paper, leads to the following system of n linear algebraic equations for deriving the values of the function $F(\bar{t})$ at the discrete set of points $\bar{t}_k = \cos[\pi(2k-1)/2n]$, $k = 1, 2, \ldots n$:

$$\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1}{\bar{t}_k-\bar{x}_r} + \pi K(\bar{x}_r,\bar{t}_k)\right]F(\bar{t}_k) = -\frac{p_0}{\mu_1\check{\mu}_{21}}, \tag{46}$$

$$\frac{\pi}{n}\sum_{k=1}^{n} F(\bar{t}_k) = 0, \quad \text{where } \bar{x}_r = \cos\frac{\pi r}{n}, \quad r = 1, 2, \ldots, n-1. \tag{47}$$

The order of the system may be reduced by one half if we take n as an odd number $n = 2n_1$ and employ the symmetry property $F(-\bar{t}) = -F(\bar{t})$ which follows from (39), (44). Defining in addition

$$F_1(\bar{t}) = -\frac{\mu_1 \breve{\mu}_{21}}{p_0} F(\bar{t}) \tag{48}$$

we obtain finally

$$\frac{1}{2n_1} \sum_{k=1}^{n_1} F_1(\bar{t}_k) \left\{ \frac{2\bar{t}_k}{\bar{t}_k^2 - \bar{x}_r^2} + \pi[K(\bar{x}_r, \bar{t}_k) - K(\bar{x}_r, \bar{t}_k)] \right\} = 1, \quad r = 1, 2, \ldots, n_1. \tag{49}$$

Note that the evaluation of the coefficients of the system in accordance with (37) and (41) does not lead to any numerical difficulties, owing to the exponential decaying (42) of the function $R(\bar{\alpha})$. The rate of the decay is defined by the distance $h_{\min}$ between the crack plane and the nearest interface. Consequently, for small values of this parameter, the region of numerical integration in (41) was correspondingly increased. Having defined the values of the function $F_1(\bar{t})$ at the n_1 points $\bar{t}_k = \cos \pi(2k-1)/(4n_1)$, $k = 1, 2, \ldots, n_1$ of the interval $[0, 1]$, it is easy to obtain the value of the function at any point of the interval by simple interpolation. For all the numerical examples considered it was sufficient to take $n_1 = 20$ to get an accuracy of three significant digits. After deriving $F_1(\bar{t})$ it is possible to find the displacements and the stresses in any cell of the layered composite by the use of (12), (32), (25), (34), (44) and (48).

3. CRACKS IN A PERIODICALLY LAYERED BIMATERIAL COMPOSITE

In this section the solution procedure developed is implemented numerically for the specific case of a periodically layered bimaterial composite. Two extremal crack positions are considered: a crack in the middle of a layer and an interface crack. The fracture properties of the composite are examined by means of the stress intensity factor defined as

$$K = \lim_{x \to a} \sqrt{2\pi(x-a)}\, \tau_1^{(0)}(x, 0). \tag{50}$$

Employing the general relations between the stresses and the crack opening displacements in the crack tip vicinity it is possible to express it through the dislocation density as following (see Erdogan and Gupta (1971)):

$$K = -\frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)} \lim_{x \to a} \sqrt{2\pi a} \left(1 - \frac{x}{a}\right)^{1/2} f(x). \tag{51}$$

Employing (44) and (48) gives

$$K = p_0 \sqrt{\pi a} F_1(1). \tag{52}$$

To get the non-dimensional quantity it is convenient to use the value of the stress intensity factor for a uniformly loaded crack in a homogeneous isotropic plate $K_{\text{hom}} = p_0 \sqrt{\pi a}$ (see Sih (1965)). Consequently, the non-dimensional stress intensity factor is given by

$$\tilde{K} = \frac{K}{K_{\text{hom}}} = F_1(1). \tag{53}$$

3.1. *Crack in the middle of a layer*

Consider a periodically layered composite body consisting of layers of the two different types defined by the thicknesses h_1, h_2 and the corresponding elastic shear moduli μ_1, μ_2. A

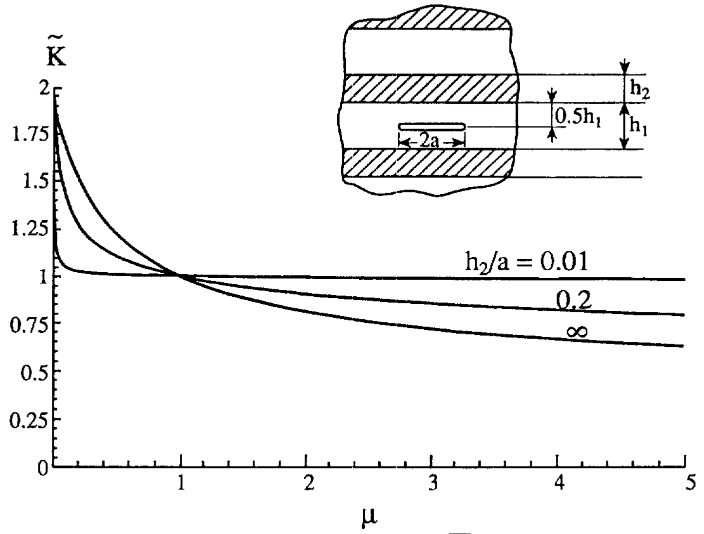

Fig. 2. Non-dimensional stress intensity factor $\tilde{K} = K(p_0\sqrt{\pi a})^{-1}$ vs shear moduli ratio $\mu = \mu_2/\mu_1$ for the case of a crack in the midplane of the layer of a periodically layered bimaterial composite with $h_1/a = 0.2$.

crack is located at the midplane of one of the layers (see insert in Fig. 2). To obtain this case from the general problem with the three types of layers it is sufficient to assume that the properties of the layers of the first and the second types are identical. Consequently, the expression for the function $R(\bar{\alpha})$ defining the Fredholm kernel (41) is found from (34), (37) after integration in ϕ as follows:

$$R(\bar{\alpha}) = -\frac{1}{2} + \frac{c_1(\bar{\alpha})}{2\sqrt{c_2(\bar{\alpha})}},$$

$$c_1(\bar{\alpha}) = (1+\bar{\mu}d_1)^2 - d_2^2(\bar{\mu}+d_1)^2,$$

$$c_2(\bar{\alpha}) = (1-\bar{\mu}^2d_1^2)^2 + (1-\bar{\mu}^2d_2^2)^2 + (1+d_1^2d_2^2)^2 + 2d_1^2d_2^2(\bar{\mu}^4 - \bar{\mu}^2d_1^2 - \bar{\mu}^2d_2^2 - 32\mu_0^2) - 2, \quad (54)$$

$$\text{where} \quad \bar{\mu} = \frac{-1+\mu}{1+\mu}, \quad \mu = \frac{\mu_2}{\mu_1}, \; \mu_0 = \frac{\mu}{(1+\mu)^2} \quad (55)$$

$$\text{and} \quad d_i = \exp\left(-\bar{\alpha}\frac{h_i}{a}\right), \; i = 1, 2. \quad (56)$$

Note the difference in the meaning of the parameter h_1 here and in the general case of the three-material composite. Substituting this expression in (41) and solving system (49) yields, in accordance with (53), the value of the non-dimensional stress intensity factor. We will denote further the layers of the first type, one of which contains the crack, as inner ones and the layers of the second type as outer layers. Let us assume that the geometry and the elastic properties of the inner layers, as well as the crack size, are fixed and view the variation of the non-dimensional parameters of the problem μ, h_2/a as changes in the characteristics of the outer layers. The influence of these parameters on the stress intensity factor is illustrated in Fig. 2. The family of graphs $\tilde{K}(\mu)$ for different thicknesses of the outer layers $h_2/a = 0.01, 0.2, \infty$ with $h_1/a = 0.2$ is exhibited. One can observe that the general behavior of the stress intensity factor is similar to that in the case of a semi-infinite crack in a periodically layered bimaterial composite (Ryvkin, 1996) and in the case of a finite crack in a three layered body (Chen and Sih, 1971). Namely, for $\mu > 1$ with the increasing shear modulus or thickness of the outer layers material, when the two layered half-spaces sandwiching the cracked layer become more rigid, the stress intensity factor decreases monotonically. When the outer layers are weaker ($\mu < 1$), increasing their thickness leads

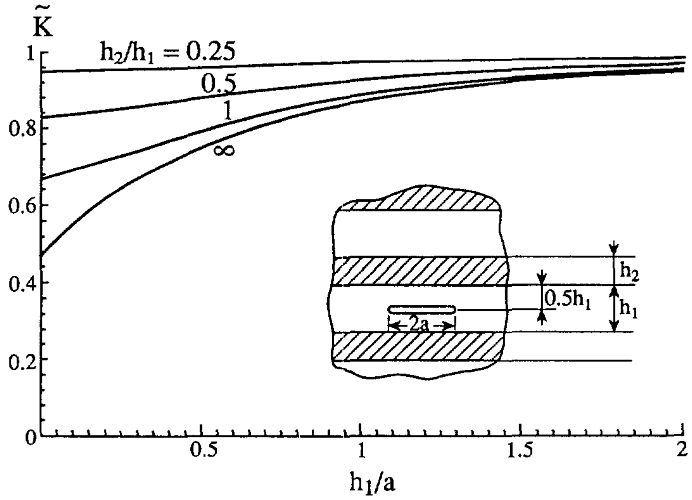

Fig. 3. Influence of the ratio of the cracked layer thickness to the crack length on the non-dimensional stress intensity factor $\tilde{K}$ for the case of the crack in the midplane of the layer. The shear moduli ratio $\mu_2/\mu_1 = 5$.

to weakening of the half-spaces and this influences the stress intensity factor in the opposite way.

For the limiting cases when the outer layer disappears ($h_2/a \to 0$) or their material becomes the same as the inner one ($\mu \to 1$) the layered composite degenerates into a homogeneous body, and $\tilde{K} \to 1$. The limiting case when $h_2 \to \infty$ yields the problem of a crack in a layer sandwiched between two homogeneous half-spaces. Consequently, the corresponding curve in Fig. 2 coincides with the graph obtained for this problem by Chen and Sih (1971). Since the cases $\mu \to 0$ and $\mu \to \infty$ correspond to the problems of a crack in a layer with free and clamped edges respectively, independently of the thickness of the outer layers, all the curves in Fig. 2 have the same limiting points for these cases. The limiting values of $\tilde{K}$ were obtained numerically by Chen and Sih (1971) in the problem of the three-layered composite and found to be $\tilde{K} = 1.98$ for the case of the free edges and $\tilde{K} = 0.36$ for the case of clamping; the analytical result for the case of the clamped edges $K = p_0\sqrt{\mu \tanh u^{-1}}$, $u = h_1/(\pi a)$ is given by Slepyan (1990). It is interesting to note that even in the case considered of a rather long crack $h_1/a = 0.2$, the change in the stress intensity factor resulting from the variation of the outer layers properties in the wide region $0.25 < \mu < 5$, $0 < h_2/h_1 < \infty$ does not exceed 50%. For shorter cracks this influence will be obviously less pronounced.

The numerical results illustrating the influence of the relative crack length on the stress intensity factor are presented in Fig. 3. Four curves $\tilde{K}(h_1/a)$ for different thicknesses ratio $h_2/h_1 = 0.25, 0.5, 1, \infty$ with $\mu = 5$ are exhibited. With a decrease of the crack length a the perturbation in the stress state generated by the crack tends to be localized in the cracked layer; consequently the normalized stress intensity factor tends to be the limiting value for a homogeneous plane $\tilde{K} = 1$. On the other hand, when the crack length increases a monotonic decrease of the stress intensity factor is observed. This results from the increasing influence of the more rigid outer layers. In the case of a crack in more rigid layer ($\mu < 1$) the observed trend will be reversed and the corresponding curves will be situated in the domain $\tilde{K} > 1$. The rate of this decrease enlarges with the growth of the outer layers thickness.

The limiting values

$$\tilde{K}_0 = \lim_{h_1/a \to 0} \tilde{K}, \tag{57}$$

corresponding to the case of a semi-infinite crack, can be evaluated analytically in the following manner. Let a tend to infinity. For a sufficiently large a one can distinguish in the stress distribution in front of the crack two zones with a square root type singular behavior. Namely, $0 < x \ll h_1$ where the near stress intensity factor K_n (denoted in the present paper as K) is dominant and $h_1 \ll x \ll a$ where the remote stress field can be approximated by the use of the far stress intensity factor K_f. It is clear that the remote field will be asymptotically the same as in the problem of a long crack in the homogeneous anisotropic body possessing the effective elastic properties of the considered laminate. But the stress intensity factor for a uniformly loaded crack in a homogeneous anisotropic body is the same as in an isotropic one $K_{an} = K_{hom} = p_0\sqrt{\pi a}$ (Sih and Chen, 1981). From dimensional considerations, it follows that when the crack length increases, the dominant zone of the above mentioned stress intensity factor will also increase and it will be valid also for the remote field. So, $K_f = p_0\sqrt{\pi a}$ and the limiting value of the non-dimensional stress intensity factor defined by (53), (57) can be expressed as

$$\tilde{K}_0 = \frac{K_n}{K_f}. \tag{58}$$

When the length a of the uniformly loaded crack in the layered composite considered tends to infinity, the form of the stress distribution in front of the crack tends to that given by the eigensolution for the semi-infinite crack located at the midplane of the cracked layer (more details about the eigensolutions of this type may be found in Ryvkin *et al.* (1995)). Consequently, to obtain the sought value $\tilde{K}_0$, it is sufficient to calculate the ratio of the near stress intensity factor to the far one for the eigen-problem. We now compare this eigen-problem with the eigen-problem for a different layered body with a semi-infinite crack considered by Ryvkin (1996). The only difference between the problems is the double thickness of the cracked layer for the second one. The form of the stress distribution in front of the crack for these problems is, of course, different but the ratio of the near stress intensity factor to the far one is the same. This can be proved by calculating the value of energy release rate through the near and the remote stress fields. Then, since the elastic properties of the bodies in the crack tip vicinity, as well as the effective elastic properties, are identical, one will get the same ratio K_n/K_f for both problems (a similar calculation for different crack position will be carried out in the next subsection). The formula for this ratio in the problem with the cracked layer of double thickness was derived by Ryvkin (1996). In view of the above reasoning this result gives the desired expression for the limiting values of the non-dimensional stress intensity factor

$$\tilde{K}_0 = \left[\frac{\mu h_1 + h_2}{\mu(h_1 + \mu h_2)}\right]^{1/4}. \tag{59}$$

It is possible to show that this formula is valid not only for a crack in the midplane of the layer but also for a non-symmetrically positioned crack.

There is also another way to interpret the dependence of the stress intensity factor upon the ratio h_1/a observed in Fig. 3. Assume that under some conditions cracks of some size a may be generated in the midplanes of the layers of the periodically layered composite. The question is to what extent can the designer influence the stress intensity factor by changing the overall thickness $h = h_1 + h_2$ of the representative cell while keeping at the same time the thicknesses ratio h_2/h_1 and, consequently, the volume fraction constant? In other words, how does the dispersion of the materials in the composite influence its fracture properties? For the case considered, when the crack is located in the softer layer ($\mu = \mu_2/\mu_1 = 5$), the qualitative answer is obvious: the increasing of dispersion (decreasing of h) will diminish the stress intensity factor. The behavior observed in Fig. 3 yields also a quantitative result. Namely, the maximum relative decrease of the stress intensity factor by increasing the dispersion of the materials is restricted by the value $1 - \tilde{K}_0$. For example, in the case of equal layers thicknesses $h_2/h_1 = 1$, the decrease will be no more than 30%.

In a number of investigations dedicated to the analysis of fracture and damage of multilayered composite materials, the local stress state in the crack vicinity is studied by

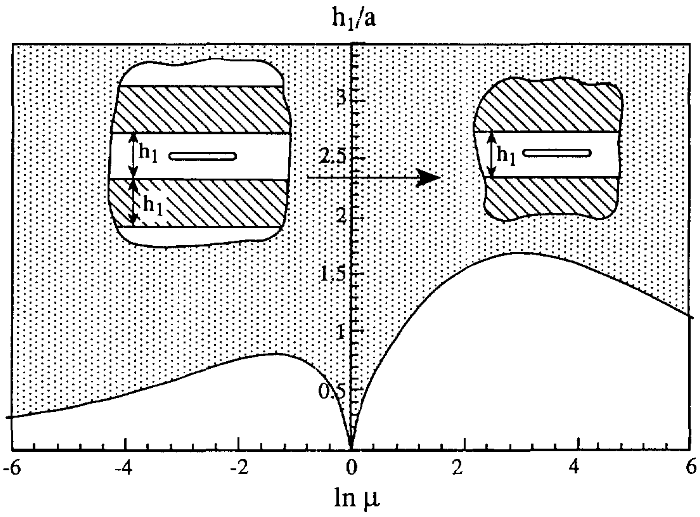

Fig. 4. Region of validity (shaded) of the sandwich approximation for the periodically layered bimaterial composite with $h_2/h_1 = 1$.

the use of the simplified model of a single crack in a layer sandwiched between two half-spaces with the elastic properties associated with the properties of the layers neighboring to the cracked one. It seems interesting to determine for which parameter combinations this simplification is correct. In the specific case of a periodically layered composite, the solution obtained allows one to get the precise answer. A reasonable equivalence criterion between multilayered and sandwich models in fracture analysis is the coincidence of the corresponding stress intensity factors within the accepted accuracy.

The numerical results for the bimaterial multilayered composite with $h_1 = h_2$ are exhibited in Fig. 4. The shaded domain corresponds to the parameter combinations for which the replacement of the periodically layered composite by the three layered system leads to less than 1% relative error. When the shear modulus of the outer layer becomes very large or negligibly small ($\ln \mu \to \pm\infty$) the influence of their thickness on the stress state near the crack tip diminishes. Therefore, the above mentioned replacement becomes valid even for long cracks and the permissible region for the parameter h_1/a extends. The same trend is observed, not surprisingly, for the limiting case of homogeneous material ($\ln \mu \to 0$). The overall observation of the numerical data leads to the following conclusion: if the thickness of the cracked layer of the periodically layered composite satisfies the inequality $h_1/a > 1.8$, the composite is equivalent to the three-layered 'sandwich' system, in the above mentioned sense, independently of the ratio of the shear moduli of the layers.

3.2. *An interface crack*

The problems of interface cracks related to the delamination phenomenon are of permanent interest in fracture analysis of multilayered composites. Consider the same bimaterial body treated in the previous subsection with a crack located not at the midplane of the layer but at the interface between the layers (see insert in Fig. 5). For such a composite even the answer on the question about the general trends in the influence of the problem parameters on the stress intensity factor is not obvious. The study of this issue, given below, will reveal some new interesting qualitative effects.

To obtain this case from the initial problem with the three types of layers we take identical properties of the layers of the second and the third types. Consequently, the function $R(\bar{\alpha})$ derived from (34), (37) is found to be

$$R(\bar{\alpha}) = -\frac{\mu}{1+\mu}\left(1+\frac{c_3(\bar{\alpha})}{\sqrt{c_4(\bar{\alpha})}}\right), \tag{60}$$

$$c_3(\bar{\alpha}) = d_1^2 d_2^2 + \bar{\mu}(d_2^2 + d_1^2) - 1, \tag{61}$$

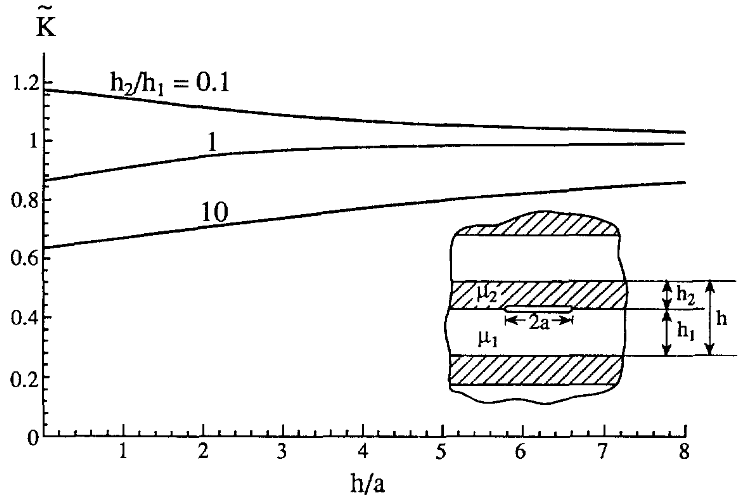

Fig. 5. Influence of the ratio of the representative cell thickness $h = h_1+h_2$ to the crack length on the non-dimensional stress intensity factor $\tilde{K}$ for the case of an interface crack.

$$c_4(\bar{\alpha}) = \bar{\mu}^4(d_1^2+d_2^2)^2+(1+d_1^2d_2^2)^2-2\bar{\mu}(d_1^2+d_2^2)(1+d_1^2d_2^2)-64\mu_0 d_1^2 d_2^2. \tag{62}$$

Here the definitions for the functions d_i, μ, $\bar{\mu}$, μ_0 remain the same as in (55), (56).

The numerical results depicting the influence of the crack length on the stress intensity factor are presented in Fig. 5. Three graphs $\tilde{K}(h/a)$ for the composites with different thicknesses ratio of the layers $h_2/h_1 = 0.1$, 1, 10 are exhibited (recall that $h = h_1+h_2$ is the overall representative cell thickness). The case of the more rigid layers of the second type $\mu = \mu_2/\mu_1 = 5$ is considered. Note that in view of the symmetry of the problem

$$\tilde{K}\left(\mu, \frac{h}{a}, \frac{h_1}{h_2}\right) = \tilde{K}\left(\frac{1}{\mu}, \frac{h}{a}, \frac{h_2}{h_1}\right). \tag{63}$$

Therefore the same graphs supply also the information about the case $\mu = 0.2$. The existence of the common asymptote $\tilde{K} = 1$ for $h/a \to \infty$ for all the curves is in accordance with the fact that the stress intensity factor for the Mode 3 crack at the interface between two half-spaces is the same as for the crack in a homogeneous body (see, for example, Sih and Chen (1981)). A similar type of behavior was observed by Chen and Sih (1971) in the problem of an interface crack between two layers of equal thickness bonded to the two dissimilar half-spaces. But in contrast to that study the stress intensity factor in the problem considered behaves monotonically and can approach unity from above as well as from below depending upon the combination of the parameters.

The result obtained, namely that the stress intensity factor for the uniformly loaded interface crack in a multilayered body may exceed the stress intensity factor for the same crack in a homogeneous space, is backed up by calculation of the limiting values $\tilde{K}_0$ for $h/a \to 0$. For this purpose, in order to use formula (58), which is valid also for the interface crack in the case considered, we have to evaluate the ratio of the near stress intensity factor to the far one in the eigensolution for the corresponding semi-infinite crack. While in the previous subsection it was possible to employ the result already obtained in another paper, here we will carry out the evaluation explicitly following Ryvkin *et al.* (1995). Consider a semi-infinite interface crack in the periodically layered two-material composite loaded at infinity. The energy release rate for the generated eigensolution, calculated through the local field near the crack tip, is related to the near stress intensity factor as follows

$$G = \frac{K_n^2}{4}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right). \tag{64}$$

The same quantity can also be calculated through the remote field. To this end, as it was noted in the previous subsection, it is possible to replace the layered bimaterial space by a homogeneous transversely isotropic medium possessing the effective elastic properties. The values of the axial and the tangential effective elastic shear moduli

$$\mu_A^* = \frac{\mu(h_1+h_2)}{\mu h_1 h_2}\mu_1, \quad \mu_T^* = \frac{h_1+\mu h_2}{h_1+h_2}\mu_1 \tag{65}$$

were derived by Postma (1955). Employing now the expression for the energy release rate for a crack in a homogeneous anisotropic material given in Sih *et al.* (1965), we obtain

$$G = \frac{K_f^2}{2\mu_1}\sqrt{\frac{h_2+\mu h_1}{\mu(h_2\mu+h_1)}}. \tag{66}$$

Note that this value for the energy release rate was obtained by Kaczynsky and Matysiak (1988) after the homogenization of the two-material periodically layered composite. Equating the right hand parts of (64) and (66) yields the desired expression for the ratio K_n/K_f. Consequently, in accordance with (58), the limiting values of the non-dimensional stress intensity factor are given by

$$\tilde{K}_0 = \left[\frac{4(\mu h_1+h_2)}{(2+\mu+\mu^{-1})(h_2\mu+h_1)}\right]^{1/4}. \tag{67}$$

It is seen that $\tilde{K}_0$ is less than one for any μ only in the case of $h_1 = h_2$. Therefore, since $\tilde{K}(h/a)$ is found to be a monotonic function one concludes that the stress intensity factor for the interface crack in a periodically layered bimaterial composite will always be less than the corresponding value for a crack in a homogeneous body only in the specific case of equal thicknesses of the layers. In the general case, using (67) it is easy to distinguish the two domains in the space of the parameters $(\mu, h_2/h_1)$ for which the limiting values for the infinite long cracks $\tilde{K}_0$ and, consequently, the values for the finite length cracks $\tilde{K}$ will be either more or less than one. Namely (recall that it is assumed that $\mu = \mu_2/\mu_1 > 1$),

$$\tilde{K} > 1 \quad \text{if } h_2/h_1 < h^*, \tag{68}$$

$$\text{and} \quad \tilde{K} < 1 \quad \text{if } h_2/h_1 > h^*, \tag{69}$$

$$\text{where} \quad h^* = \frac{3\mu+1}{\mu(\mu+3)}. \tag{70}$$

It is worthwhile to study the mentioned effect also by examining the dependence $\tilde{K}(\mu)$. Three curves for $h_2/h_1 = 0.1, 0.25, 1$ with $h/a = 0.5$ are depicted in Fig. 6. Only the curve associated with the equal thicknesses of the layers $h_2/h_1 = 1$ is symmetric and does not cross the line $\tilde{K} = 1$ corresponding to the case of the homogeneous body. For the case of different thicknesses one notices that if the thinner layers are more rigid, then there is always a certain region of shear moduli ratio $1 < \mu < \mu^*$, for which the stress intensity factor for the interface crack is more than for the crack in the homogeneous body. The relations (68)–(70) confirm this numerical result mathematically: it is seen that if μ tends to unity from above then h* approaches unity from below. On the other hand, when $\mu \to \infty$, $\tilde{K}$ is always less than unity. The observed non-monotonic behavior of the stress intensity factor can also be confirmed in the following manner. Consider a periodically layered bimaterial composite with $\mu_2/\mu_1 > 1$ and $h_2/h_1 < 1$. For a moderate shear moduli ratio the more rigid thin layer adjacent to the crack may be viewed as a flexible rod which passes the loading to the crack tip, consequently enlarging the stress intensity factor relative to the case of the homogeneous space. But if the thin layers become very rigid, then the problem of the soft layer with clamped edges and boundary debonding emerges. This problem, from symmetry

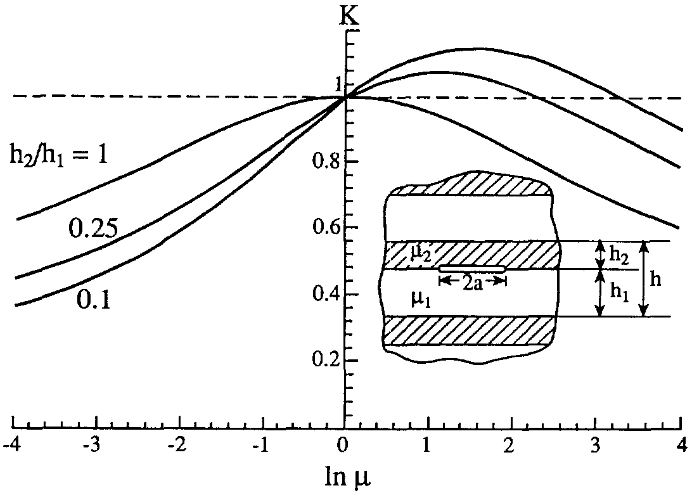

Fig. 6. Non-dimensional stress intensity factor $\tilde{K}$ vs shear moduli ratio $\mu = \mu_2/\mu_1$ for the case of the interface crack in a periodically layered bimaterial composite with $h/a = 0.5$. The dashed line corresponds to the case of the homogeneous body.

considerations, is equivalent to the problem of the layer of double thickness with the clamped edges and the midplane crack for which the stress intensity factor is less than for the crack in the homogeneous space (see previous subsection).

4. CONCLUDING REMARKS

The technique employed in the paper for the solution of the problem of the Mode 3 crack in a periodically layered three-material composite is found to be a convenient tool for the fracture analysis of multilayered bodies. It may also be applied for the mathematically more complicated in-plane fracture modes. The combined use of the representative cell and the dislocation methods allows one to reduce the problem for the periodically layered body to the integral equation without any approximating homogenization procedure. Consequently, the stress strain state in each point of the body can be derived with high accuracy. This is particularly important for the application of the fracture criterion based on the local stress distribution.

For the specific case of the periodically layered bimaterial composite, the fracture analysis was carried out by a parametric study of the stress intensity factor. For the crack positioned at the midplane of the layer, the general trends in the stress intensity factor behavior as a function of the shear moduli ratio of the layers and the crack length, are found to be, as expected, the same as in the simple model of a cracked layer sandwiched between two half spaces. The comparison of the values of the stress intensity factors for the periodically layered composite and the corresponding three layered sandwich system led to limits of validity for the approximate sandwich model. It appears, for example, that if in the periodically layered composite with equal thicknesses of the layers the layer thickness is more than 90% of the crack length, then this composite can be replaced by the three layered system with less than 1% error in the stress intensity factor. Moreover, this is true independently of the shear moduli ratio of the layers. The results obtained offer an opportunity to estimate analytically the influence of the material dispersion (the ratio of the overall cell thickness $h_1 + h_2$ to the crack length) on the stress intensity factor.

The monotonic behavior of the stress intensity factor as a function of the relative crack length is observed not only for the case of the crack in the layer but also for the case of the interface crack. Note that in the consideration of an interface crack in the four layered system (Chen and Sih, 1971) the essentially different non-monotonic behavior has been recovered. This may be explained by the significant influence of the overall anisotropic properties of the composite on the stress intensity factor, especially for the case of long cracks. Therefore, simplifying the problem for a multilayered composite by replacing part of the body with the homogeneous isotropic domain must be carried out very carefully.

The simple formula for the values of the normalized stress intensity factor in the limiting case of infinite long crack have been derived. It is shown, using this result, that the stress intensity factor for both cases of a crack in the layer and an interface crack may be either more or less than the corresponding value for the crack in a homogeneous space. Being trivial for the first case, this fact is somewhat unexpected regarding the crack at the interface. It appears that if the more rigid layers of the periodically layered bimaterial composite are thinner than the softer ones, then increasing their rigidity may lead either to a decrease or to an increase of the stress intensity factor for the interface crack, depending upon the specific combination of the parameters. This effect may be important in the design of new layered composite materials.

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